# MATH 2050 - Limit Theorems for functions

(Reference: Bartle §4.2)

(iff statement)

Motto: By Sequential Criteria, we get limit theorems for functions from the corresponding limit theorems for sequences

Recall: For sequences, $(x_n)$ convergent $\Rightarrow$ $(x_n)$ bdd

We have a similar result for functions.

**Boundedness Thm:**

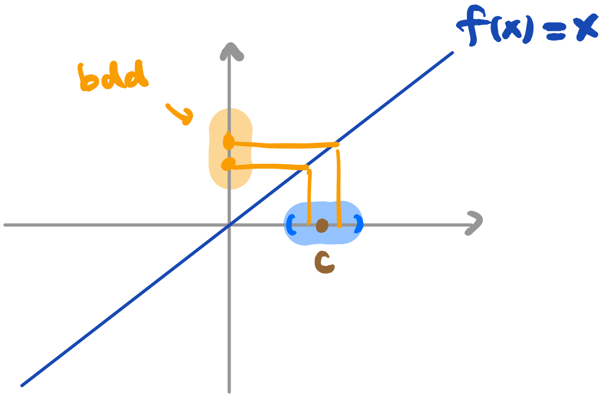
$\lim_{x \to c} f(x)$ exists $\Rightarrow$ $f$ is "bdd in a neighborhood of $c$"

i.e. $\exists M > 0$ and $\exists \delta > 0$ s.t.

(Note: $\Leftarrow$ not true) $\quad |f(x)| \leq M \quad \forall\ |x - c| < \delta$ and $x \in A$

Remark: $f$ may not be bdd "globally".

E.g.)



Proof: By $\varepsilon$-$\delta$ def$^{n}$, $\lim_{x \to c} f(x) = L$ $\Rightarrow$ Take $\varepsilon = 1$

Then $\exists\ \delta = \delta(1) > 0$ s.t. $|f(x) - L| < \varepsilon = 1$

whenever $x \in A$ and $0 < |x - c| < \delta$

by Δ-ineq

$$\Rightarrow \begin{cases} |f(x)| \leq |f(x) - L| + |L| < 1 + |L| \\ \text{whenever } x \in A \text{ and } 0 < |x - c| < \delta \end{cases}$$

If we take $M := \max\{1 + |L|, \underbrace{|f(c)|}_{\text{if } c \in A}\} > 0$, then

we have $|f(x)| \leq M \quad \forall\, x \in A$ s.t $|x - c| < \delta$

$\square$

<u>Def<sup>n</sup></u>: Given $f, g : A \to \mathbb{R}$ functions defined on the same $A$,

then we can define new functions:

- $(f \pm g)(x) := f(x) \pm g(x)$ $\qquad f \pm g : A \to \mathbb{R}$
- $(fg)(x) := f(x)\, g(x)$ $\qquad fg : A \to \mathbb{R}$
- $\left(\frac{f}{g}\right)(x) := \dfrac{f(x)}{g(x)}$ $\qquad \frac{f}{g} : A \setminus \{x \in A \mid g(x) = 0\} \to \mathbb{R}$

<u>Thm</u>: (1) $\lim\limits_{x \to c} (f \pm g)(x) = \lim\limits_{x \to c} f(x) \pm \lim\limits_{x \to c} g(x)$

(2) $\lim\limits_{x \to c} (fg)(x) = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x)$

**(3) $\lim\limits_{x \to c} \left(\frac{f}{g}\right)(x) = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$

extra careful!

provided that $\lim\limits_{x \to c} f(x)$, $\lim\limits_{x \to c} g(x)$ exist,

and for (3), additionally, we need $\lim\limits_{c \to c} g(x) \neq 0$

**Examples:** $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$ provided $c \neq 0$ ; $\lim_{x\to 2} \frac{x^3-4}{x+1} = \frac{4}{3}$

$$\lim_{x\to 2} \frac{x^2-4}{3x-6} = \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{3\cancel{(x-2)}} = \frac{4}{3}.$$

**Proof of (2):** IDEA: Use seq criteria.

Take $(x_n)$ in $A$ s.t. $x_n \neq c$ $\forall n \in \mathbb{N}$ and $\lim(x_n) = c$.

Seq criteria $\Rightarrow$ $(f(x_n)) \to \lim_{x\to c} f(x)$ & $(g(x_n)) \to \lim_{x\to c} g(x)$

Limit Thm for seq. $\Rightarrow$ $(f(x_n)\cdot g(x_n)) \to \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

Seq criteria $\Rightarrow$ $\lim_{x\to c} (fg)(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

$\square$

## Squeeze / Sandwich Thm (for functions)

Let $g, f, h : A \to \mathbb{R}$ be functions s.t.

$$g(x) \leq f(x) \leq h(x) \quad \forall x \in A \quad \cdots\cdots (\dagger)$$

Suppose $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$.

THEN. $\lim_{x\to c} f(x) = L$.

**Remarks:** 1) The existence of $\lim_{x\to c} f(x)$ is a conclusion

2) One only requires $(\dagger)$ to hold in some neighborhood of $c$.

**Proof:** Use sequential criteria.

Let $(x_n)$ be a sequence in $A$ s.t. $x_n \neq c$ $\forall n \in \mathbb{N}$. $\lim(x_n) = c$.

**Claim:** $\lim f(x_n) = L$

**Pf:** By $(\dagger)$, $g(x_n) \le f(x_n) \le h(x_n)$ $\forall n \in \mathbb{N}$

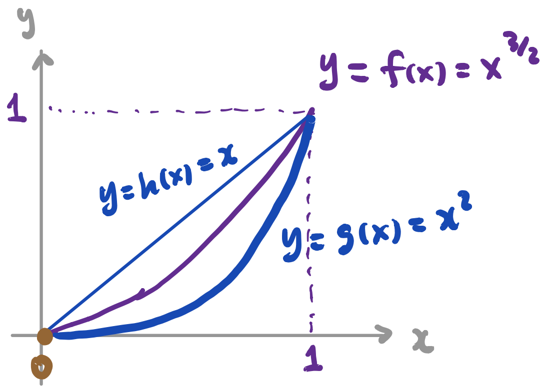
By seq. criteria, $\lim g(x_n) = L = \lim h(x_n)$.

By Squeeze Thm for seq., $\lim f(x_n) = L$.

□

**Example 1:** $\boxed{\lim_{x\to 0} x^{3/2} = 0}$

**Proof:** Here: $f : A := \{x \in \mathbb{R} \mid x \ge 0\} \to \mathbb{R}$ where $f(x) := x^{3/2}$.



Take $g, h : A \to \mathbb{R}$ as

$g(x) = x^2$ & $h(x) = x$.

Note that

$$x^2 \le x^{3/2} \le x \qquad \forall x \in [0,1]$$

By squeeze thm. since

$$\lim_{x\to 0} x^2 = 0 = \lim_{x\to 0} x$$

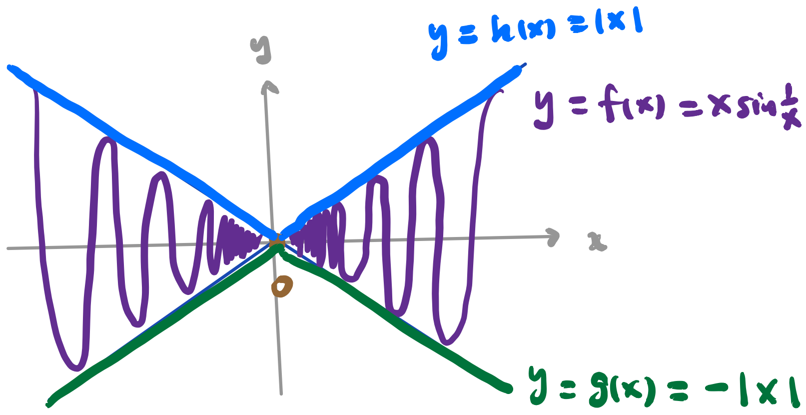
So $\lim_{x\to 0} x^{3/2} = 0$.

□

## Example 2: $\boxed{\lim_{x\to 0} x\sin\frac{1}{x} = 0}$

( Recall: $\lim_{x\to 0}(\sin\frac{1}{x})$ DOES NOT EXIST by seq. criteria. )

**Proof:** Here: $f: A = \mathbb{R}\setminus\{0\} \to \mathbb{R}$, and $f(x) = x\sin\frac{1}{x}$.



Since $|\sin\frac{1}{x}| \le 1$, we have

$$-|x| \le x\sin\frac{1}{x} \le |x| \quad \forall x\in A$$

Now $\lim_{x\to 0}|x| = 0 = \lim_{x\to 0} -|x|$

By Squeeze Thm.

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

□

**Prop:** Suppose $\lim_{x\to c} f(x) = L > 0$. THEN, $\exists\, \delta > 0$ s.t.

$$f(x) > 0 \quad \forall\, x\in A \text{ s.t. } 0<|x-c|<\delta$$

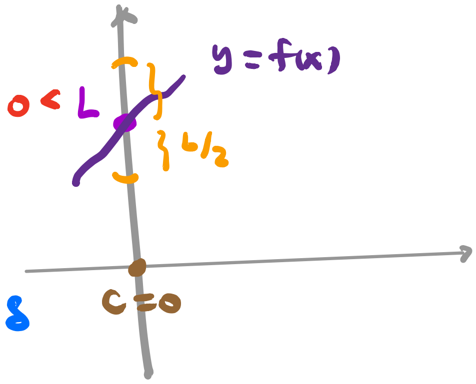
**Remark:** The Prop. DOES NOT hold if we replace $>$ by $\ge$.

e.g. $L = 0$ (see Example 2 above)

**Proof:** Use $\varepsilon$-$\delta$ def$^n$!

Take $\varepsilon := L/2 > 0$.

Then $\exists\, \delta = \delta(\frac{L}{2}) > 0$ s.t.



$$|f(x) - L| < \varepsilon = \frac{L}{2} \quad \forall\, 0<|x-c|<\delta$$

$$\Rightarrow \quad f(x) > L - \frac{L}{2} = \frac{L}{2} > 0 \quad \forall\, 0<|x-c|<\delta$$

□